Mid term exam

Section I

State whether the following statements are True or False.

(i) Every system of Equations have a solution. F

(ii) The Row space and Column space of matrix A have same dimension. T

(iii) The Norm of a unit vector is zero. F

(iv) Every linearly dependent set contains the zero vector. F

(v) An Inconsistent systemhas more than one solution. F

Section II

From the following choose the correct answer:

1) The system of linear equation 4x-2y=1, 16x-8y=4, has

(a) Unique solution (b)infinitely many solution (c) no solution (d)None of these

2) If $\boldsymbol{u}$= (1, 1, 2) and $\boldsymbol{v}$= (1, 0, 2) then the value of $\boldsymbol{u}.\boldsymbol{v}$ is

(a)-5 (b)6 (c)5 (d)4

3) The inverse of the matrix $\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$=

(a)$\begin{pmatrix} -1/2 & 3/2 \\ 1 & -3 \end{pmatrix}$ (b)$\begin{pmatrix} -1 & 3 \\ 2 & -7 \end{pmatrix}$ (c)$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$ (d) None of these

4) If the Rank of the given matrix A=$\begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & 7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix}$ is 2,then Nullity (A) is

(a) 2 (b)4 (c)8 (d)None

5) If A is 3x7 matrix and B is 7x5 Matrix, then AB is
(a) 3x3 (b) 7x5 (c)3x5 (d)5x5

6) $\boldsymbol{v} = (-3,2,1)$ then the value of $\|\boldsymbol{v}\|$ is
(a)$\sqrt{13}$ (b) $\sqrt{14}$ (c)14 (d) $\sqrt{15}$

7) Which of these determinants has the value -6?

(a) $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 2 & 4 & 8 \end{vmatrix}$ (c) $\begin{vmatrix} 6 & 6 \\ 1 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 0 & 2 \\ 3 & 5 \end{vmatrix}$

8) The vectors $\boldsymbol{u_1} = (3, -1), \boldsymbol{u_2} = (4,5), \boldsymbol{u_3} = (-4,7)$ in R^n are

(a) Linearly dependent (b) Linearly independent (c) Orthogonal (d) None

Section III

Attempt all questions

1. Using row reduction to find determinant of A, where A= $\begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -1 \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$\rightarrow R_3 \rightarrow R_3 - 2R_1 = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$\rightarrow R_3 \rightarrow R_3 - 10R_2 = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (3)(-55)(1) = 165$$

2. Find a unit vector that is orthogonal to both $\boldsymbol{u}$=(1,0,1) and $\boldsymbol{v}$=(0,1,1).

$z.\boldsymbol{u}$=0

$(z_1, z_2, z_3).(1,0,1) = z_1 * 1 + z_2 * 0 + z_3 * 1 = z_1 + z_3 \rightarrow z_1 + z_3 = 0$

$z_1 = -z_3$

$z.\boldsymbol{v}$=0

$(z_1, z_2, z_3).(0,1,1) = z_1 * 0 + z_2 * 1 + z_3 * 1 = z_2 + z_3 \rightarrow z_2 + z_3 = 0$

$\rightarrow z_2 = -z_3 = z_1$

Therefore z= $(z_1, z_1, -z_1)$ *is orthogonal to both u and v.*

3. Solve by Gaussian elimination and back-substitution

$$x + y + 2z = 9$$

$$2x + 4y - 3z = 1$$

$$3x + 6y - 5z = 0$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\frac{R_2}{2} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$-2R_3 \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

By back-substitution:

z= 3

$2y - \frac{7}{2}z = -\frac{17}{2} \to 2Y - \frac{7}{2} = -\frac{17}{2} \to \boldsymbol{y = 2}$

$x + y + 2z = 9 \to X = 9 - (2) - 2(3) = 1 \to \boldsymbol{x = 1}$

4. Use Cramer's rule to solve

$$\begin{array}{rcrcrcr} x_1 & + & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}$$

$A=\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $A_1=\begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$, $A_2=\begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$, $A_3=\begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$

det (A) $=1\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 24 + 20 = 44$

det $(A_1) = 6\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 30 & 4 \\ 8 & -2 \end{vmatrix} = 144 - 184 = 40$

det $(A_2) = 1\begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} - 6\begin{vmatrix} -3 & 6 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 30 \\ -1 & 8 \end{vmatrix} = 42 + 18 + 12 = 72$

det $(A_3) = 1\begin{vmatrix} 4 & 30 \\ -2 & 8 \end{vmatrix} + 6\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$= 32+60 +6(6+4)=92+60=150

$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$

$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$

$x_3 = \frac{\det\,(A_3)}{\det\,(A)} = \frac{152}{44} = \frac{38}{11}$

8

Home work Assignment for Week 8

Section I

State whether the following statements are True (T) or False (F)

1-If λ is an eigenvalue of a matrix A, then the linear system **(λI-A)X =0** has only the trivial solution. **F**

2-If 0 is an eigenvalue of a matrix A, then A^2 is singular. **T**

3-If A is diagonalizable, then there is a unique matrix P such that $P^{-1} A P$ is diagonal. **F**

4-If A is diagonalizable, then A^T is diagonalizable. **T**

5-Every eigenvalue of a complex symmetric matrix is real. **F**

6-If A is a square matrix with distinct real eigenvalues, then it is possible to solve $X' = A X$ by diagonalization. **T**

Section II

From the following choose the correct answer

1- The eigenvalues of the following Matrix are:

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

a) 2,3 b) -1,-2,-3, c) 3 **d)** 1, 2, 3

2- The characteristic equations of the following matrix:

$$\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$$

a) $\lambda^2 - 8\lambda + 8 = 0$ b) $\lambda^2 - 8\lambda + 16 = 0$ **c)** $\lambda^2 - 2\lambda + 8 = 0$ d) $\lambda^2 + 8\lambda - 16 = 0$

3- The matrix P that diagonalizes *to A is* ,

$$A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$

a) $P = \begin{bmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{bmatrix}$ **b)** $P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$ c) $P = \begin{bmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{bmatrix}$ d) $P = \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & 1 \end{bmatrix}$

4- If u= (I, 2i, 3) and V= (4, -2i, 1+i) then u.v is

a) 1+i b) -1-i **c)** -1+i d) 1-i

5- Solve the system

$$Y_1' = Y_1 + 4Y_2$$
$$Y_2' = 2Y_1 + 3Y_2$$

a) $\mathbf{Y_1 = c_1 e^{5X} - 2c_2 e^{-X}, \; Y_2 = c_1 e^{5X} + 2c_2 e^{-X}}$

b) $Y_1 = c_1 e^{-5X} - 2c_2 e^{-X}, \; Y_2 = c_1 e^{-5X} + 2c_2 e^{-X}$

c) $Y_1 = c_1 e^{5X} - 2c_2 e^{-X}, \; Y_2 = c_1 e^{5X} - 2c_2 e^{-X}$

d) $Y_1 = c_1 e^{-5X} - 2c_2 e^{-X}, \; Y_2 = c_1 e^{-5X} - 2c_2 e^{-X}$

Section III

1) Find the eigenvalues of A9 for

$$A=\begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & .5 & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer: $(1/2)^9, 2^9$

2) Find the characteristic equations of the following matrices:

$$A=\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer: $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$

3) find a matrix P that diagonalizes A, and compute $P^{-1} A P$

$$A=\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Answer

$$P=\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^{-1} A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4) **Compute A^{11} for**

$$A=\begin{bmatrix}-1 & 7 & -1\\ 0 & 1 & 0\\ 0 & 15 & -2\end{bmatrix}$$

Answer: $\begin{bmatrix}-1 & 10237 & -2047\\ 0 & 1 & 0\\ 0 & 10245 & -2048\end{bmatrix}$

5) Find $\bar{A}, Re\,(A), Im(A), det(A), tr(A)$ for

$$A = \begin{bmatrix}-5i & 4\\ 2-i & 1+5i\end{bmatrix}$$

Answer:

$\bar{A} = \begin{bmatrix}5i & 4\\ 2+i & 1-5ii\end{bmatrix}$, Re (A) = $\begin{bmatrix}0 & 4\\ 2 & 1\end{bmatrix}$, Im (A) = $\begin{bmatrix}-5 & 0\\ -1 & 5\end{bmatrix}$

Det (A) = 17-I, Tr (A) =1

6) If u= (i, 2i, 3) and V= (4, -2i, 1+i), and w=(2-i, 2i, 5+3i) k= 2i

Find $\overline{(u.\overline{V}) - \overline{(w.u)}}$

Answer: -11-14i

7) (a) Solve the system

$$\begin{aligned} Y_1' &= 4Y_1 \qquad + Y_3\\ Y_2' &= -2Y_1 + Y_2\\ Y_3' &= -2Y_1 \qquad + Y_3 \end{aligned}$$

(b) Find the solution that satisfies the initial conditions $y_1(0)$ =-1, $y_2(0)$ =1, and $y_3(0)$ =0

Answer: a) Y_1= $-c_2\, e^{2X} + 2c_3e^{3X}$, Y_2= $c_1\, e^{X} + 2c_2e^{2X} - c3e^{3X}$, Y_3= $2c_2\, e^{2X} - c_3e^{3X}$

b) = e^{2X} $-2e^{3X}$, Y_2= e^{X} - $2e^{2X} + 2e^{3X}$, Y_3= $-2\, e^{2X} + 2e^{3X}$

Section 1

State whether the following statements are True or False

1-The inner product operation must satisfy 2 conditions.
a- True
b- False

Answer b

2-If the columns of A are linearly independent, the the equation Ax=b has exactly one least squares solution
a-True
b-False

Answer a

3- In a inner product space (V,(.,.)) if x and y are unit vectors orthogonal to each other then ìì x+y ìì=2
a-True
b-False

Answer b

4-The inner product of two vectors cannot be a negative real number
a-True
b-False

Answer b

5- if we have $\vec{u} = (4, 3), \vec{v} = (3, 5)$ then $\|\vec{v}\|$ is $\sqrt{34}$
a-True
b-False

Answer a

Section 2

1- Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on R^2
a- (0,6), (7,0)
b- (3,4),(2,6)
c- (6,9),(5,2)
d- (0,0), (0,6)
Answer d

2-if $\|u\| = \sqrt{30}$, $\|v\| = \sqrt{18}$, and $\langle u, v\rangle = -9$ so $\cos\theta$ equal
a-9
b-5
c- $-\frac{3}{2\sqrt{15}}$
d-7

Answer c

3-Find the cosine of the angle, θ, between $\boldsymbol{p} = x - x^2$, and $\boldsymbol{q} = 2 + 2x + 2x^2$

a-0

b-3

c-6

d-9

Answer a

4-Let <u, v> be the Euclidean inner product on R2, and let $\vec{u} = (4, 3)$, $\vec{v} = (3, 5)$ then $\langle \vec{u}, \vec{v} \rangle$ is

a-8

b-27

c-6

d-9

Answer b

5-A straight line is

a- y=ax+b

b-a+bx+cx2

d- a+bx+c2+dx3

d-none of the above

Answer a

Section 3

1-Compute ‹**u**, **v**› using the inner product on M_{22}.

$$\mathbf{u} = \begin{bmatrix} 9 & -8 \\ 9 & 18 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 & 9 \\ 1 & 1 \end{bmatrix}$$

Answer

‹**u**, **v**› = 9 · (− 1) + (− 8) · 9 + 9 · 1 + 18 · 1 = -54

2- Let R^3 have the Euclidean inner product. Find the cosine of the angle, θ, between $\vec{u}$=(-1,6,2) and $\vec{v}$(4,3,-5)

Answer

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\|\vec{v}\|}$$

$$= \frac{(-1)(4) + (6)(3) + (2)(-5)}{\sqrt{(-1)^2 + (6)^2 + (2)^2}\sqrt{(4)^2 + (3)^2 + (-5)^2}}$$

$$= \frac{4}{\sqrt{41}\sqrt{50}} = \frac{4}{\sqrt{2050}}$$

3- Find the least squares solution of the linear equation $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Answer
$$\vec{x} = \begin{bmatrix} \frac{651}{462} \\ \frac{-95}{110} \end{bmatrix} = \begin{bmatrix} \frac{31}{22} \\ \frac{-19}{22} \end{bmatrix}$$

4-calculate (u-2v, 3u+4v)

Answer
$$= 3\|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle - 8\|\mathbf{v}\|^2$$

5- Apply the Gram Schmidt process to transform the basis vectors u1=(1,1,1), u2=(0,1,1), u3=(0,0,1), into a orthogonal basis(v1,v2,v3)

Answer v1=(1,1,1), v2=(-2/3,1/3,1/3), v3=(0, -1/2,1/2)

6- Find the least squares solution of the linear equation
x1-x2=4
3x1+2x2=1
-2x1+4x2=3

7- Find the error

Answer
6- x1=17/95, x2=143/285

7-error=4556

Section 1:

State whether the following statements are True or False:

1- If a square matrix A is orthogonal, then $A^{-T} = A$. True
2- If A is a square matrix, and $\det(A) = 2$, then A is not orthogonal. True
3- If S is an orthogonal basis for n-dimensional inner product space V, then V is the Euclidean inner product space. True
4- A square matrix whose rows form an orthogonal set is orthogonal . False
5- An 3×2 matrix A is orthogonal if $A^T A = I$. False
6- Every orthogonal matrix is orthogonally diagonalizable. False
7- If A is orthogonally diagonalizable, then A has real eigenvalues. True

Section 2:

Choose the correct answer:

1- One of the following matrices is positive definite:

a- $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ b- $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ c- $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ d- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

2- If A is a square orthogonal symmetric matrix , then:

a- $\det(A) = 2$ b- $A^{-1}.A = A^2$ c- $tr(A) > 0$ d- A is not invertible

3- One of the following matrices is orthogonally diagonalizable:

a- $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$ b- $\begin{bmatrix} 3 & 2 & -4 \\ 2 & 4 & 6 \\ -4 & 6 & -1 \end{bmatrix}$

c- $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ d- $\begin{bmatrix} 3 & 5 & 6 \\ 7 & -1 & 4 \\ 3 & 5 & 0 \end{bmatrix}$

4- One of the following quadratic forms is classified as Indefinite:

a- ${x_1}^2 - {x_2}^2$ b- ${x_1}^2 + {x_2}^2$ c- $(x_1 - x_2)^2$ d- $-{x_1}^2 - 3{x_2}^2$

Section 3:

1- Determine which of the following matrices are orthogonal. For those that are orthogonal, find the inverse.

a- $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$: orthogonal, the inverse is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

b- - $\begin{bmatrix} 3 & -2 & 1 \\ 4 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$: not orthogonal.

c- - $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$: orthogonal, the inverse is - $\begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

2- Find the characteristic equation of the given symmetric matrices, and then determine the dimension of the eigenspaces.

a- $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} : \lambda^2 - 5\lambda = 0\,, \quad \lambda = 0: one\ dimensional, \lambda = 5: one\ dimensional.$

b- $\begin{bmatrix} 3 & 5 & 6 \\ 7 & -1 & 4 \\ 3 & 5 & 0 \end{bmatrix} : \lambda^3 - 27\lambda - 54 = 0, \lambda = 6: one\ dimensional, \lambda = -3: two\ dimensional$.

c- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} : \lambda^3 - 3\lambda^2 = 0, \lambda = 3: one\ dimensional, \lambda = 0: two\ dimensional$.

3- Find a matrix P that orthogonally diagonalizes A. And determine $P^{-1}AP$.

$$A = \begin{bmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{-2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix} \quad , P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}.$$

4- Express the quadratic form in the matrix notation $x^T Ax$, where A is a symmetric matrix.

a- $3{x_1}^2 + 7{x_2}^2$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$b- 9{x_1}^2 - {x_2}^2 + 4{x_3}^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3.$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(11)

Saudi Electronic University (college of IT).

MATH 251.

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Saudi Electronic University

2011 1432

Linear Algebra
Assignment
Week11

Chapter 8

Section I

State whether the following statements are True or False

1. If A is an 3×5 matrix and T is a transformation defined by T(x)=Ax, then the domain of T is $\mathbb{R}^3$. **False**
2. A linear transformation preserves the operations of vector addition and scalar multiplication. **True**
3. If L is a linear operator mapping a vector space V into a vector space W, then $L(\mathbf{0}_V)=\mathbf{0}_W$. **True**
4. The range of L is the image of the entire vector space. **True**
5. If A and B are the same size and both represent the same linear operator, they are similar. **True**

Section II

From the following choose the correct answer

1. Let T be a linear transformation from R^n to R^m and let $\vec{u} = T(\vec{0})$ where $\vec{0}$ is the zero vector in R^n. Choose the correct statement
   A) $\vec{u}$ is a zero vector in R^n
   B) $\vec{u}$ is a zero vector in R^m if and only if $n \le m$
   C) $\vec{u}$ is a zero vector in R^m if and only if $n = m$
   D) **$\vec{u}$ is a zero vector in R^m**

2. Let A be an $n \times n$ matrix of rank m. Any matrix similar to A:

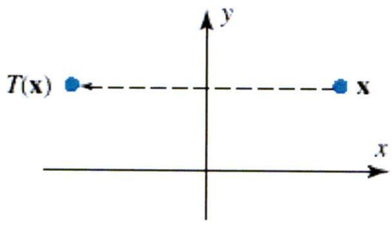
   A) may have rank $\le$ n
   B) may have rank $\ge$ m
   C) may have any rank $\ge$ m and $\le$ n
   D) **must have rank m**

3. Determine whether the linear transformation T is one-to-one.

$$T{:}R^m \rightarrow R^n,\ n < m.$$

A) The answer depends upon the value of $m - n$
B) T is one-to-one
C) ***T*** **is not one-to-one**
D) it is impossible to determine whether T is one-to-one

4. As indicated in the accompanying figure, let $T{:}R^2 \rightarrow R^2$ be the linear operator that reflects each point about the y-axis.



Find the kernel of T. Is T one-to-one?

A) ker(T) = {(0, y) | where y is any real number} ; T is one-to-one
B) ker(T) = {(x, 0) | where x is any real number} ; T is not one-to-one
C) **ker(T) = {0} ; T is one-to-one**
D) ker(T) = {**0**} ; T is not one-to-one

5. Find the domain and codomain of $T_2 \circ T_1$, and find $(T_2 \circ T_1)(x_1, x_2)$.

$$T_1(x, y) = (2x, 4y),\ T_2(x, y) = (x - y, x + y)$$

A) The domain and codomain of $T_2 \circ T_1$ are R^3, and $(T_2 \circ T_1)(x_1, x_2) = (2x_1 - 4x_2, 2x_1 + 4x_2)$
B) **The domain and codomain of $T_2 \circ T_1$ are R^2, and $(T_2 \circ T_1)(x_1, x_2) = (2x_1 - 4x_2, 2x_1 + 4x_2)$**
C) The domain and codomain of $T_2 \circ T_1$ are R^3, and $(T_2 \circ T_1)(x_1, x_2) = (2x_1 - 2x_2, 4x_1 + 4x_2)$
D) The domain and codomain of $T_2 \circ T_1$ are R^2, and $(T_2 \circ T_1)(x_1, x_2) = (2x_1 - 2x_2, 4x_1 + 4x_2)$

Section III

1. Let T be multiplication by the matrix A, find the rank and nullity of T.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & -4 \\ -1 & 8 & -16 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & -4 \\ -1 & 8 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix}$$

R(T) is two-dimensional, so rank(T) = 2.
ker(T) is one-dimensional, so nullity (T) = 1.

2. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, 0)$, and let $T{:}R^2 \rightarrow R^2$ be the linear operator such that

$$T(\mathbf{v}_1) = (1, -4) \text{ and } T(\mathbf{v}_2) = (-8, 1)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(9, -5)$.

Sol:

$T(x_1, x_2) = x_2T(\mathbf{v}_1) + (x_1 - x_2)T(\mathbf{v}_2) = x_2(1, -4) + (x_1 - x_2)(-8, 1) =$
$(-8x_1 + 9x_2, x_1 - 5x_2)$

From this formula, we obtain $T(9, -5) = (-117, 34)$.

3. Find ker(T), and determine whether the linear transformation T is one-to-one.

$$T(x, y) = (x, y, 2x + 6y)$$

Sol:

ker(T) = (0, 0)

So the transformation is one-to-one.

4. Let $T_1: V \rightarrow V$ be the dilation $T_1(\mathbf{v}) = 2\mathbf{v}$. Find a linear operator $T_2: V \rightarrow V$ such that $T_1 \circ T_2 = I$ and $T_2 \circ T_1 = I$.

Sol:

$$T_2(\mathbf{v}) = \frac{1}{2}\mathbf{v}.$$

5. Let $T: R^2 \rightarrow R^2$ be multiplication by

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Determine whether T has an inverse; if so, find

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

Sol:

The matrix T has an inverse, and the standard matrix for T^{-1} is

$$\left[T^{-1}\right] = [T]^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

Expressing this result in horizontal notation yields

$$T^{-1}(x_1, x_2) = (x_1 - 2x_2, -2x_1 + 5x_2)$$

6. Let $T_1: R^2 \rightarrow R^2$ and $T_2: R^2 \rightarrow R^2$ be the linear operators given by the formulas

$$T_1(x, y) = (x + y, x - y) \text{ and } T_2(x, y) = (10x + y, x - 10y)$$

Find formulas for $T_1^{-1}(x, y)$, and $T_2^{-1}(x, y)$, and $(T_2 \circ T_1)^{-1}(x, y)$.

Sol:

$$\left[T_1^{-1}\right] = [T_1]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\left[T_2^{-1}\right] = [T_2]^{-1} = \begin{bmatrix} \frac{10}{101} & \frac{1}{101} \\ \frac{1}{101} & -\frac{10}{101} \end{bmatrix}$$

$$\left[(T_2 \circ T_1)^{-1}\right] = [T_2 \circ T_1]^{-1} = \begin{bmatrix} \frac{11}{202} & -\frac{9}{202} \\ \frac{9}{202} & \frac{11}{202} \end{bmatrix}$$

It follows that:

$$T_1^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left[T_1^{-1}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{1}{2}y \end{bmatrix}$$

$$T_2^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left[T_2^{-1}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{10}{101} & \frac{1}{101} \\ \frac{1}{101} & -\frac{10}{101} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{10}{101}x + \frac{1}{101}y \\ \frac{1}{101}x - \frac{10}{101}y \end{bmatrix}$$

$$(T_2 \circ T_1)^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left[(T_2 \circ T_1)^{-1}\right]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{202} & -\frac{9}{202} \\ \frac{9}{202} & \frac{11}{202} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{202}x - \frac{9}{202}y \\ \frac{9}{202}x + \frac{11}{202}y \end{bmatrix}$$

7. Let $T{:}P_2 \rightarrow P_1$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) - (19a_1 + 6a_2)x$$

Find the matrix for T with respect to the standard bases $B = \{1, x, x^2\}$ and $B' = \{1, x\}$ for P_2 and P_1.

Sol:

The matrix for T with respect to B and B' is

$$[T]_{B',B} = \left[[T(\mathbf{u}_1)]_{B'} \ \ [T(\mathbf{u}_2)]_{B'} \ \ [T(\mathbf{u}_3)]_{B'} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -19 & -6 \end{bmatrix}$$

Section (1):

State whether the following statements are True (T) or False (F)

(1) If a square matrix *A* has an *LU*-decomposition, then *A* has a unique *LDU*-decomposition. **True**

(2) Every square matrix has an *LU*-decomposition. **False**

(3) If *A* is an m □ n matrix, then A^TA is an m □ m matrix **False**

(4) If *A* is an m □ n matrix, then the eigenvalues of A^TA are positive real numbers. **False**

(5) Every m □ n matrix has a singular value decomposition. **True**

Section (2):

Choose the correct answer from the following:

(1) Which of the following sets of eigenvalues have a dominant eigenvalue.

(a) {-4, -3,4, 1} **(b)** {-3, -1, 0, 2}

(c) { 0, 3, -3, -2} (d) {-5, 3, -2, 5}

(2) The approximation of the time required to the forward phases of Gauss-Jordan elimination equal:

(a) $2/3n^3$ (b) n^2 (c) $2n^3$

(3) If A is an m □ n matrix, then A and A^TA have the same:

(a) Null space (b) row space

(d) rank **(d)** All of them

(4) The singular values of A= $\begin{matrix} 5 & 0 \\ 0 & 2 \end{matrix}$

(a) {0, 1} **(b)** { 2, 5} (c) {2,0} {0,5}

Section(3)

(1) Find an *LU*-decomposition of A

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$$

Answer:

	Reduction to Row Echelon form	Row operation	Elementary Matrix Corresponding to the Row operation	Inverse of the Elem. Matrix
	$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$			
step 1		$\frac{1}{2} \times$ row 1	$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$	$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
	$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix}$			
step 2		row 1 + row 2	$E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$
	$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$			
step 3		$\frac{1}{3} \times$ row 2	$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$	$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$
	$U = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$			

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

so, $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

(2) Apply the power method with Euclidean scaling to the matrix A, starting with x_o and stopping at x_4 . Compare the resulting approximations to the exact values of the dominant eigenvalue and the corresponding unit eigenvector.

$$A = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}; \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Answer:

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix}; \ \mathbf{x}_2 \approx \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix}; \ \mathbf{x}_3 \approx \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix}; \ \mathbf{x}_4 \approx \begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix};$$

dominant eigenvalue: $\lambda = 2 + \sqrt{10} \approx 5.16228;$

dominant eigenvector: $\begin{bmatrix} 1 \\ 3 - \sqrt{10} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.16228 \end{bmatrix}$

(3) Approximate the time required to execute the forward phase of Gauss–Jordan elimination for a system of 100,000 equations in 100,000 unknowns using a computer that can execute 1 gigaflop per second. Do the same for the backward phase.

Answer

Gigaflops for forward phase $\approx 2/3\ n^3$ □ 10^{-9} =

$2/3\ (10^5)^3$ □ $10^{-9} = 6.67$ □ 10^5

Gigaflops for backward phase $\approx n^2$ □ 10^{-9} = 10 s

(4) find the distinct singular values of $A = \begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix}$

Answer:

$$A^T A = \begin{matrix} 1 & 2 \\ -2 & 1 \end{matrix} \; \square \; \begin{matrix} 1 & -2 \\ 2 & 1 \end{matrix} = \begin{matrix} 5 & 0 \\ 0 & 5 \end{matrix}$$

The characteristic polynomial of $A^T A$ is $(\lambda I - A) = (\lambda - 5)^2$

so the eigenvalue of $A^T A$ is $\lambda = 5$ and the singular value of A is $\sqrt{5}$

Q 1: Choose the correct answer:

1)The point (3,0) satisfy one of the following systems:

a) $x + y \geq 5$, $x + 2y \geq 3$

b) $3x - y \geq 9$, $4x + 5y \leq 11$

c) $12x - y \geq 35$, $3x + 4y \leq 10$

d) $2x + y \geq 6$, $3x - 5y \geq 15$

2) one of the following system is bounded:

a) $y \geq x$, $y \geq -x$

b) $y \leq x+3$, $y \leq 4-x$

c) $x \geq 1$, $y \geq 3$

d) $y \leq 2x+3$, $y \leq 6 - x$, $y \geq 2$

3) the point at which $f = 3x + 5y$ has the highest value is:

a) (0,2) b) (4,0) c) (3,1) **d) (2,5)**

4) one of the following triples is the solution to the linear programming

max $2x_1 + 3x_2 + 2x_3$ subject to $\begin{cases} x_1 + 4x_2 \le 4 \\ x_1 - x_2 + 3x_3 \le 5 \end{cases}$, $x_1, x_2, x_3 \ge 0$

a) (0,1,2) b) (4,0,0.5) **c) $(4,0,\frac{1}{3})$** d) (1,0,1)

5) One of the following is a valid objective function for a linear programming problem:

a) Max (5xy)

b) Min(4x + 3y + (2/3)z)

c) Max ($5x^2$ + 6y)

d) Min((x + y)/z)

6) The Slack is :

a. the difference between the left and right sides of a constraint.

b. the amount by which the left side of a ≤ constraint is smaller than the right side.

c. the amount by which the left side of a ≥ constraint is larger than the right side.

d. exist for each variable in a linear programming problem.

7) To find the optimal solution to a linear programming problem using the graphical method

a. find the feasible point that is the farthest away from the origin.

b. find the feasible point that is at the highest location.

c. find the feasible point that is closest to the origin.

d. None.

8) Infeasibility means that the number of solutions to the linear programming models that satisfies all constraints is:

a. at least 1.

b. 0.

c. an infinite number.

d. at least 2.

Q 2: State whether the following statements are True or False:

1) In linear programming problems, a linear objective function that is to be maximized or minimized. **True**

2) All variables in linear programming problems restricted to nonnegative values. **True**

3) The maximization or minimization of a quantity is the objective of linear programming. **True**

4) The following LP problem has an unbounded feasible region:

Minimize $c = x - y$
subject to $4x - 3y \leq 0$
$x + y \leq 10$
$x \geq 0, y \geq 0$

False

5) Every minimization problem can be converted into a standard maximization problem. **False**

Q 3:

a)Find the maximum values of P=3x+2y subject to

$x + 4y \leq 20$

$2x + 3y \leq 30$

$x \geq 0, y \geq 0$

Answer: (15,0)

b)Find the minimum values of C=6x+8y subject to

$40x + 10y \geq 2400$

$10x + 15y \geq 2100$

$5x + 15y \geq 1500$

$x \geq 0, y \geq 0$

Answer: (30,120).

Q 4:

In the following matrices, find the pivot element if simplex is not done. Find the values of all the variables if simplex is done.

a)

x	y	u	v	p	
1	1	1	0	0	10
2	1	0	1	0	20
-3	2	0	0	1	0

Answer: Pivot on row1 column 1 or row 2 column 1(you can choose both)

b)

x	y	u	v	p	
-1	1	1	0	0	0
-2	1	0	1	0	20
-3	2	0	0	1	0

Answer:

NO SOLUTION (nothing to pivot on and there is a negative number in the bottom row).

Q 5:

Solve the following system graphically:

$-3x + y \leq 5$

$3x + y \leq 5$

$y \geq 3.$

5. Solve the following system graphically:

$-3x + y \leq 5$

$3x + y \leq 5$

$y \geq 3$.

Answer:

$-3x + y \leq 5$ $\qquad$ $y = 5 + 3x$

x	0	-1
y	5	2

$3x + y \leq 5$ $\qquad$ $y = 5 - 3x$

x	0	1
y	5	2

$y \geq 3$.